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Note

On graphs with equal domination and independent domination numbers

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Abstract

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Allan and Laskar have shown that $K_{1,3}$-free graphs are graphs with equal domination and independent domination numbers. In this paper new classes of graphs with equal domination and independent domination numbers are presented. In particular, the result of Allan and Laskar is generalized.

Our notation generally follows that used in [3]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. If $X \subseteq V(G)$ then $[X]$ is the induced subgraph of G with the vertex set X. For a vertex x of G, $N(x)$ denotes the set of all vertices adjacent to x in G and $\bar{N}(x) = N(x) \cup \{x\}$. More generally, $N(X) = \bigcup_{x \in X} N(x)$ and $\bar{N}(X) = N(X) \cup X$ for a non-empty subset X of $V(G)$.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N(v) \cap D \neq \emptyset$ for every $v \in V(G) - D$. A set $I \subseteq V(G)$ is an *independent set* of G if $N(v) \cap I = \emptyset$ for every $v \in I$. A set $I \subseteq V(G)$ is an *independent dominating set* of G if I is both an independent and dominating set. The following proposition characterizes minimal dominating sets in graphs.

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Proposition [6]. *A dominating set D of a graph G is minimal if and only if for each $d \in D$ either* (i) $N(d) \cap D = \emptyset$ *or* (ii) *there exists $c \in V(G) - D$ such that $N(c) \cap D = \{d\}$.*

A set $X \subseteq V(G)$ is called an *irredundant set* of G if for each $x \in X$ either (i) $N(x) \cap X = \emptyset$ or (ii) there exists $y \in V(G) - X$ such that $N(y) \cap X = \{x\}$. The cardinality of a minimum dominating set of a graph G is called the *domination number* of G and is denoted by $\gamma(G)$. The cardinality of a minimum independent dominating set of a graph G is called the *independent domination number* of G and is denoted by $i(G)$. Finally, the minimum cardinality of a maximal irredundant set of a graph G is called the *irredundance number* of G and is denoted by $\mathrm{ir}(G)$.

The above definitions and the proposition imply that $\mathrm{ir}(G) \leq \gamma(G) \leq i(G)$ for every graph G. Hedetniemi and Mitchell [5] proved that the line graph $L(T)$ of a tree T satisfies $\gamma(L(T)) = i(L(T))$. Allan and Laskar [1] showed that if G does not have an induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$. From this they deduced that for any graph G, $\gamma(L(G)) = i(L(G))$ and $\gamma(M(G)) = i(M(G))$, where $L(G)$ and $M(G)$ is the line graph and the middle graph of G, respectively. Recently Harary and Livingston [4] characterized trees T with $\gamma(T) = i(T)$. Favaron [2] has proved that, if G contains no induced subgraph isomorphic to $K_{1,3}$ or to H (with the vertex set $V(H) = \{v_1, \ldots, v_7\}$ and the edge set $E(G) = \{v_iv_{i+1}: i = 1, \ldots, 6\} \cup \{v_3v_5\}$), then $\mathrm{ir}(G) = \gamma(G) = i(G)$. The aim of this paper is to present new classes of graphs with equal domination and independent domination numbers. In particular, the main result of [1] is generalized.

The main result of this paper is the following theorem.

Theorem. *If a graph G contains no induced subgraph isomorphic to one of the graphs $H_1, \ldots, H_{16}$ in Fig. 1, then $\gamma(G) = i(G)$.*

Proof. Assume that none of the graphs $H_1, \ldots, H_{16}$ is an induced subgraph of G. We will show that $\gamma(G) = i(G)$. Since $\gamma(G) \leq i(G)$, it is sufficient to prove that in G there is a minimum dominating set which is independent, that is, there exists an independent dominating set of the cardinality $\gamma(G)$. Suppose to the contrary that each minimum dominating set of G is not independent. Let D_0 be a minimum dominating set of G such that $e([D_0])$ is the minimum number taken over all minimum dominating sets of G, where $e([X])$ denotes the number of edges in the subgraph induced by $X \subseteq V(G)$. Take two adjacent vertices x_1, x_2 from D_0 and the sets

$$I_i = \{v \in V(G) - D_0: N(v) \cap D_0 = \{x_i\}\} \quad (i = 1, 2),$$

and

$$I_{1,2} = \{v \in V(G) - D_0: N(v) \cap D_0 = \{x_1, x_2\}\}.$$

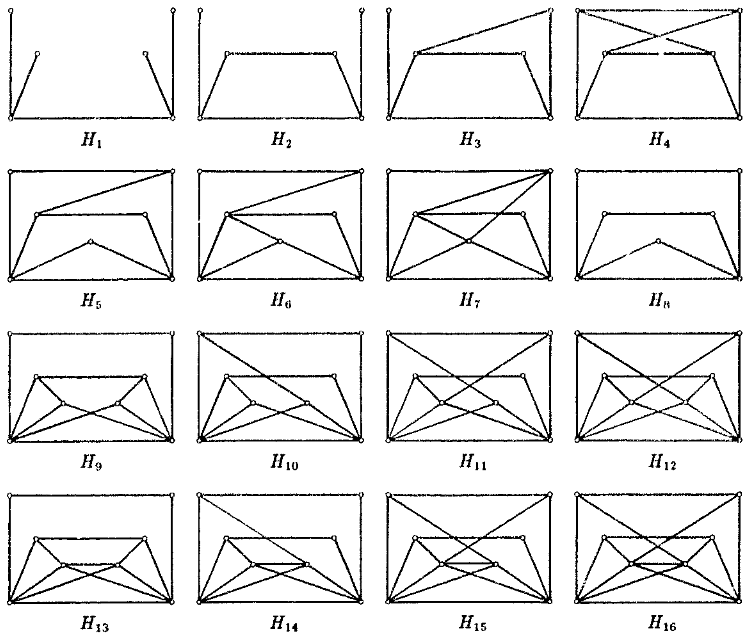

Fig. 1.

Since every minimum dominating set is minimal, it follows from Proposition that the sets I_1, I_2 are nonempty and disjoint. We derive contradictions in two cases.

Case 1: *For* $i = 1$ *or* 2, *there exists a vertex* $v_i \in I_i$ *such that* $I_i \subset \bar{N}(v_i)$.

Then, it is easy to see, the set $D_1 = (D_0 - \{x_i\}) \cup \{v_i\}$ (for $i = 1$ or 2) is a minimum dominating set of G and $e([D_1]) < e([D_0])$, contradicting the choice of D_0.

Case 2: *For* $i = 1, 2$ *and every* $y \in I_i$, $I_i \not\subset \bar{N}(y)$.

Then in I_i $(i = 1, 2)$ there are non-adjacent vertices. Let v_1, v_2 and u_1, u_2 be non-adjacent vertices from I_1 and I_2, respectively. From the fact that the subgraph $[\{x_1, x_2, v_1, v_2, u_1, u_2\}]$ is not isomorphic to H_4 it follows that there exist $v \in \{v_1, v_2\} \subseteq I_1$ and $u \in \{u_1, u_2\} \subseteq I_2$ such that $vu \notin E(G)$.

We now claim that $I_1 \cup I_2 \subset \bar{N}(\{v, u\})$ if $v \in I_1$, $u \in I_2$ and $vu \notin E(G)$. For if not, then there exist vertices $v_0 \in I_1$ and $u_0 \in I_2$ such that $v_0u_0 \notin E(G)$ and the set $(I_1 \cup I_2) - \bar{N}(\{v_0, u_0\})$ is not empty. Without loss of generality we may assume that $I_1 - \bar{N}(\{v_0, u_0\}) \neq \emptyset$. Take any vertex $\bar{v}$ from $I_1 - \bar{N}(\{v_0, u_0\})$ and any vertex $\bar{u}$ from $I_2 - \bar{N}(u_0)$. Then, since $x_1x_2, x_1v_0, x_1\bar{v}, x_2u_0, x_2\bar{u} \in E(G)$ and v_0u_0, $v_0\bar{v}$, $u_0\bar{u}$, $u_0\bar{v} \notin E(G)$, the induced subgraph $[\{x_1, x_2, v_0, \bar{v}, u_0, \bar{u}\}]$ of G is isomorphic to one of the graphs H_1, H_2, H_3, a contradiction. This contradiction shows that $I_1 \cup I_2 \subset \bar{N}(\{v, u\})$ whenever $v \in I_1$, $u \in I_2$ and $vu \notin E(G)$.

Next we show that there exist vertices $v_0 \in I_1$, $u_0 \in I_2$ such that $v_0u_0 \notin E(G)$ and $I_{1,2} \subset N(\{v_0, u_0\})$. Suppose to the contrary that the set $I_{1,2} - N(\{v, u\})$ is not empty for every $v \in I_1$, $u \in I_2$ if $vu \notin E(G)$. It is easy to see that for non-adjacent

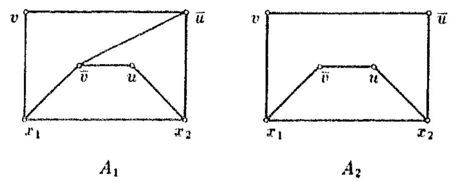

Fig. 2.

vertices $v \in I_1$, $u \in I_2$ and for any vertices $\bar{v} \in I_1 - \bar{N}(v)$ and $\bar{u} \in I_2 - \bar{N}(u)$, the subgraph $A = [\{x_1, x_2, v, \bar{v}, u, \bar{u}\}]$ is isomorphic to one of the graphs A_1, A_2 in Fig. 2, as otherwise A would be isomorphic to one of the forbidden graphs H_1, H_2, H_3. We distinguish two subcases.

Subcase 2.1: *A is isomorphic to* A_1.

Then for any $x \in I_{1,2} - N(\{v, u\})$, the subgraph $[V(A) \cup \{x\}]$ is isomorphic to H_{5+i} if $|\{\bar{v}, \bar{u}\} \cap N(x)| = i$ $(i = 0, 1, 2)$, a contradiction.

Subcase 2.2: *A is isomorphic to* A_2.

First let us observe that if there exists a vertex $x \in I_{1,2} - (N(\{v, u\}) \cup N(\{\bar{v}, \bar{u}\}))$, then the subgraph $[V(A) \cup \{x\}]$ is isomorphic to H_8, contradicting the hypothesis of the theorem. Thus assume that the set $I_{1,2} - (N(\{v, u\}) \cup N(\{\bar{v}, \bar{u}\}))$ is empty. Since the sets $I_{1,2} - N(\{v, u\})$ and $I_{1,2} - N(\{\bar{v}, \bar{u}\})$ are not empty and $I_{1,2} \subset N(\{v, u\}) \cup N(\{\bar{v}, \bar{u}\})$, the sets $(I_{1,2} - N(\{v, u\})) \cap N(\{\bar{v}, \bar{u}\})$ and $(I_{1,2} - N(\{\bar{v}, \bar{u}\})) \cap N(\{v, u\})$ are non-empty and disjoint. For $y \in (I_{1,2} - N(\{v, u\})) \cap N(\{\bar{v}, \bar{u}\})$ and $z \in (I_{1,2} - N(\{\bar{v}, \bar{u}\})) \cap N(\{v, u\})$ we consider the subgraph $[V(A) \cup \{y, z\}]$. It is easy to see that $[V(A) \cup \{y, z\}]$ is isomorphic to one of the graphs $H_9, \ldots, H_{12}$ $(H_{13}, \ldots, H_{16}$, resp.) if $yz \notin E(G)$ $(yz \in E(G)$, resp.). Again, we have obtained a contradiction to the hypothesis of the theorem and we therefore henceforth suppose that there exist vertices $v \in I_1$, $u \in I_2$ such that $vu \notin E(G)$ and $I_{1,2} \subset N(\{v, u\})$.

The proof may now be completed. It follows from the above established observations that there exist vertices $v \in I_1$, $u \in I_2$ such that $vu \notin E(G)$ and $I_1 \cup I_2 \cup I_{1,2} \subset \bar{N}(\{v, u\})$. Then consider the set $D_1 = (D_0 - \{x_1, x_2\}) \cup \{v, u\}$. Let $x \in V(G) - D_1 = P \cup R$, where

$$P = V(G) - (D_0 \cup I_1 \cup I_2 \cup I_{1,2}) \quad \text{and} \quad R = (I_1 \cup I_2 \cup I_{1,2} \cup \{x_1, x_2\}) - \{v, u\}.$$

The fact that D_0 is a dominating set of G and the definitions of the sets I_1, I_2, and $I_{1,2}$ imply that $N(x) \cap (D_0 - \{x_1, x_2\}) \neq \emptyset$ and, therefore, $N(x) \cap D_1 \neq \emptyset$ for each $x \in P$. From the choice of the vertices v and u we have $N(x) \cap \{v, u\} \neq \emptyset$ for each $x \in R$. Hence D_1 is a dominating set of G. Since $|D_1| = |D_0|$ and $N(\{v, u\}) \cap D_1 = \emptyset$, D_1 is a minimum dominating set of G with $e([D_1]) < e([D_0])$. Again, we have obtained a contradiction with the choice of D_0. This contradiction completes the proof. □

From the previous theorem the following results are obtained.

Corollary 1. *Let G be a graph without triangles. If G contains no induced subgraph isomorphic to one of the four graphs H_1, H_2, H_3, H_4 in Fig.* 1, *then* $\gamma(G)=i(G)$.

Proof. If G is without triangles, then clearly it contains no induced subgraph isomorphic to one of the graphs $H_5, \ldots, H_{16}$ (see Fig. 1) and the result follows from Theorem. □

Corollary 2. *If a graph G has no induced subgraph isomorphic to one of the six graphs H_1, H_2, H_3, H_4, A_1, A_2 (see Figs.* 1 *and* 2), *then* $\gamma(G)=i(G)$.

Proof. The result is immediate from Theorem, since G does not have an induced subgraph isomorphic to one of the graphs $H_5, \ldots, H_{16}$ if it does not have an induced subgraph isomorphic to A_1 or A_2. □

Corollary 3. *If G is a graph in which no two induced subgraphs isomorphic to $K_{1,3}$ have a common edge and different centers, then* $\gamma(G)=i(G)$.

Proof. Under these conditions on G, none of the graphs $H_1, \ldots, H_{16}$ is an induced subgraph of G and the result follows from Theorem. □

The next two results are immediate from Corollary 3.

Corollary 4 [1]. *If G is a graph which does not have an induced subgraph isomorphic to $K_{1,3}$, then* $\gamma(G)=i(G)$.

For the next corollary we need the following definition. The subdivision graph $S(G)$ of a graph G is a graph with the property that a one-to-one correspondence can be made between its vertices and the elements of G such that two vertices of $S(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. In other words, $S(G)$ is a graph obtained from G by inserting a new vertex on each edge of G.

Corollary 5. *For any graph G, $\gamma(S(G))=i(S(G))$, where $S(G)$ is the subdivision graph of G.*

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